Elliptic solutions of the Toda chain and a generalization of the Stieltjes-Carlitz polynomials

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Abstract

We construct new elliptic solutions of the restricted Toda chain. These solutions give rise to a new explicit class of orthogonal polynomials which can be considered as a generalization of the Stieltjes-Carlitz elliptic polynomials. The recurrence coefficients and the weight function of these polynomials are expressed explicitly. In the degenerated cases of the elliptic functions the modified Meixner polynomials and the Krall-Laguerre polynomials appear.

1. Restricted Toda chain and orthogonal polynomials

The main purpose of the present paper is explicit construction of orthogonal polynomials which are a generalization of the famous Stieltjes-Carlitz orthogonal polynomials connected with elliptic functions. For theory of these polynomials and history of their discovery see, e.g. [15], [7], [25].

Our main tool will be connection between orthogonal polynomials depending on an additional "time" parameter t and solutions of the so-called restricted Toda chain. This connection allows us first to construct some explicit "elliptic" solutions of the Toda chain and then to reconstruct corresponding orthogonal polynomials and their orthogonality measure. As we will see, the Stieltjes-Carlitz polynomials appear to be a very special case of these constructed orthogonal polynomials corresponding to a "zero time" $t = 0$ case. We derive also explicit recurrence coefficients $u_n(t), b_n(t)$ for the obtained orthogonal polynomials. These coefficients are expressed in terms of the Weierstrass elliptic functions in n and t . Recall that the recurrence coefficients of the Stieltjes-Carlitz polynomials are expressed in terms of linear and quadratic polynomials in n. We show that obtained polynomials are orthogonal on the whole real axis with a positive discrete measure. The measure is constructed explicitly. It appears that the parameter t can take values only inside of some interval (so-called admissible interval) in order for the measure and the recurrence coefficients will be well defined. The Stieltjes-Carlitz case corresponds to the middlepoint $t = 0$ of this interval.

We recall basic definitions and results concerning relations between Toda chain and orthogonal polynomials [21], [4], [17].

The Toda chain equations are [22]

$$\dot{u}_n = u_n(b_n - b_{n-1}), \quad \dot{b}_n = u_{n+1} - u_n \tag{1.1}$$

with additional condition

$$u_0 = 0 \tag{1.2}$$

where the dot indicates the differentiation with respect to t. In what follows we will call equations (1.1) with restriction (1.2) *the restricted Toda chain* (TC) equations.

Let $P_n(x; t)$ be orthogonal polynomials satisfying the three-term recurrence relation

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x) \tag{1.3}$$

with initial conditions

$$P_0 = 1, \quad P_1(x) = x - b_0. \tag{1.4}$$

We will assume that $u_n \neq 0, n = 1, 2, \ldots$. By the Favard theorem [8], there exists a nondegenerate linear functional σ such that the polynomials $P_n(x)$ are orthogonal with respect to

it:

$$\sigma(P_n(x)P_m(x)) = h_n \delta_{nm}, \tag{1.5}$$

where h_n are normalization constants. The linear functional σ can be defined through its moments

$$c_n = \sigma(x^n), \quad n = 0, 1, \ldots. \tag{1.6}$$

It is usually assumed that $c_0 = 1$ (standard normalization condition), but we will not assume this condition in the followings. So we will assume that c_0 is an arbitrary nonzero parameter.

Introduce the Hankel determinants

$$D_n = \det(c_{i+j})_{i,j=0,\ldots,n-1}, \quad D_0 = 1, \quad D_1 = c_0. \tag{1.7}$$

Then the polynomials $P_n(x)$ can be uniquely represented as [8]

$$P_n(x) = \frac{1}{D_n} \begin{vmatrix} c_0 & c_1 & \ldots & c_n \\ c_1 & c_2 & \ldots & c_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n-1} & c_n & \ldots & c_{2n-1} \\ 1 & x & \ldots & x^n \end{vmatrix}. \tag{1.8}$$

The normalization constants are expressed as

$$h_n = \frac{D_{n+1}}{D_n}, \quad h_0 = D_1 = c_0. \tag{1.9}$$

The recurrence coefficients u_n satisfy the relation

$$u_n = \frac{h_n}{h_{n-1}} = \frac{D_{n-1}D_{n+1}}{D_n^2}. \tag{1.10}$$

Thus we have

$$h_n = c_0 u_1 u_2 \cdots u_n. \tag{1.11}$$

Assume now that the polynomials $P_n(x;t)$ depend on a real parameter t through their recurrence coefficients $u_n(t)$, $b_n(t)$. Then the restricted Toda chain equations (RTE) are equivalent to the condition

$$\dot{P}_n(x;t) = -u_n P_{n-1}(x;t). \tag{1.12}$$

It is possible to choose initial moment $c_0(t)$ (normalization) such that the RTE are equivalent to the very simple condition

$$\dot{c}_n = c_{n+1}, \tag{1.13}$$

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i.e.
$$c_n(t) = \frac{d^n c_0(t)}{dt^n}. \tag{1.14}$$

Hence, for the Toda chain case, the Hankel determinants $D_n = D_n(t)$ have the form

$$D_n(t) = \det(c_0^{(i+k)}(t))_{i,k=0,\dots,n-1}, \quad D_0 = 1, \quad D_1 = c_0, \tag{1.15}$$

where $c_0^{(j)}$ means the j-th derivative of $c_0(t)$ with respect to t.

Under this condition, the RTE are equivalent also to the equations

$$\frac{d^2 \log D_n}{dt^2} = \frac{D_{n-1} D_{n+1}}{D_n^2}, \quad n = 1, 2, \dots. \tag{1.16}$$

Note also that for the Hankel determinants of the form (1.15) we have the useful relation

$$b_n = \frac{\dot{D}_{n+1}}{D_{n+1}} - \frac{\dot{D}_n}{D_n} \tag{1.17}$$

or, equivalently,

$$b_n = \dot{h}_n / h_n. \tag{1.18}$$

In particular, for $n = 0$ we have from (1.18)

$$b_0 = \frac{\dot{c}_0}{c_0}. \tag{1.19}$$

The relation (1.19) allows us to restore $c_0(t)$ if the recurrence coefficient $b_0 = b_0(t)$ is known explicitly from Toda chain solutions (1.1).

The Stieltjes function $F(z)$ is defined as a generating function of the moments [8]

$$F(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \cdots + \frac{c_n}{z^{n+1}} + \cdots. \tag{1.20}$$

If moments c_n depend on t according to the Toda Ansatz (1.13), we then have

$$\dot{F}(z;t) = \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_n}{z^n} + \cdots = zF(z) - c_0. \tag{1.21}$$

In fact, the relation (1.21) is equivalent to the restricted TC equations (1.13).

We consider also a so-called E-generating function of another type:

$$\Phi(p) = \sum_{k=0}^{\infty} c_k \frac{p^k}{k!}. \tag{1.22}$$

The relationship between functions $F(z)$ and $\Phi(p)$ is given by the (formal) Laplace transform:

$$F(z) = \sum_{k=0}^{\infty} c_k z^{-k-1} = \sum_{k=0}^{\infty} c_k \int_0^{\infty} \frac{p^k e^{-pz}}{k!} dp = \int_0^{\infty} e^{-pz} \Phi(p) dp. \tag{1.23}$$

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For the case of the RTE with condition (1.13) we see that generating function $\Phi(p)$ is given automatically by the formal Taylor expansion

$$\Phi(p;t) = \sum_{k=0}^{\infty} c_k(t)\frac{p^k}{k!} = \sum_{k=0}^{\infty} c_0^{(k)}(t)\frac{p^k}{k!} = c_0(t+p) \tag{1.24}$$

of $c_0(t+p)$. Thus the E-generating function is given just by the shifted $c_0(t+p)$ zero-moment function. The Stieltjes function is given then as the Laplace transform

$$F(z;t) = \int_0^{\infty} e^{-pz} c_0(t+p)dp. \tag{1.25}$$

It should be noted, however, that formula (1.25) has rather formal meaning. In practice, there are situations when direct application of this formula may be problematic, if, e.g. integral in rhs (1.25) diverges. As a simple example consider the case when $c_0(t)$ has the expression

$$c_0(t) = \sum_{k=-\infty}^{\infty} \mu_k \exp(\nu_k t) \tag{1.26}$$

with some complex constants μ_k, ν_k. Then

$$c_n(t) = \frac{d^n c_0(t)}{dt^n} = \sum_{k=-\infty}^{\infty} \mu_k \nu_k^n \exp(\nu_k t)$$

and the Stieltjes function $F(z;t)$ has the expression

$$F(z;t) = \sum_{n=0}^{\infty} c_n(t)z^{-n-1} = \sum_{k=-\infty}^{\infty} \frac{\mu_k \exp(\nu_k t)}{z - \nu_k} \tag{1.27}$$

i.e. the orthogonality measure in this case is located at the points ν_k of the complex plane with the corresponding concentrated masses $M_k(t) = \mu_k \exp(\nu_k t)$. Formula (1.27) is a special case of the well known result for the restricted Toda chain: if $d\rho(x)$ is an orthogonality measure for the orthogonal polynomials $P_n(z;0)$ (i.e. for initial value of time $t = 0$), then for arbitrary t the measure will be [4], [18]

$$d\rho(x;t) = const \; \exp(xt) \, d\rho(x) \tag{1.28}$$

where the constant in rhs of (1.28) is not essential and is needed only to provide the normalization condition for the measure. Formula (1.27) corresponds to the special case of the measure

$$d\rho(x) = \sum_{k=-\infty}^{\infty} \mu_k \, \delta(x - \nu_k)dx \tag{1.29}$$

On the other hand, the generating function $\Phi(p;t)$ has the expression

$$\Phi(p;t) = \sum_{n=0}^{\infty} \frac{c_n(t)p^n}{n!} = \sum_{k=-\infty}^{\infty} \mu_k \exp(\nu_k(p+t)) \tag{1.30}$$

If one applies formula (1.23) for the Laplace transform to the function $\Phi(p;t)$ given by (1.30) we get

$$F(z;t) = \sum_{k=-\infty}^{\infty} e^{\nu_k t} \mu_k \int_0^{\infty} e^{-p(z-\nu_k)} dp \tag{1.31}$$

Doing "naively" we can put

$$\int_0^{\infty} e^{-p(z-\nu_k)} dp = (z-\nu_k)^{-1} \tag{1.32}$$

in (1.31) and obtain desired formula (1.27) for the Stieltjes function $F(z;t)$. However, formula (1.32) is correct only if $Re(z) > Re(\nu_k)$. For $Re(z) < Re(\nu_k)$ we may still use the same formula (1.31) because it corresponds to the true formal series (1.27).

Sometimes the following trick will be useful. Consider the modified function $\tilde{c}_0(t) = c_0(it)$ (i.e. just pass to the "imaginary time"). Construct the Hankel determinants $\tilde{D}_n(t)$ form the new moments $\tilde{c}_n(t) = i^n c_n(it)$. Clearly, $\tilde{D}_n(t) = i^{n(n-1)} D_n(it)$. Define corresponding recurrence coefficients $\tilde{b}_n(t), \tilde{u}_n(t)$. We have

$$\tilde{b}_n(t) = ib_n(it), \quad \tilde{u}_n(t) = -u_n(it) \tag{1.33}$$

Obviously they will satisfy the restricted Toda chain equations (1.1). Construct corresponding orthogonal polynomials

$$\tilde{P}_n(z;t) = i^n P_n(z/i;it) \tag{1.34}$$

These "modified" orthogonal polynomials satisfy the recurrence relation

$$\tilde{P}_{n+1}(z;t) + \tilde{b}_n(t)\tilde{P}_n(z;t) + \tilde{u}_n\tilde{P}_{n-1}(z;t) = z\tilde{P}_n(z;t) \tag{1.35}$$

with the initial conditions

$$\tilde{P}_0(z;t) = 1, \quad \tilde{P}_1(z;t) = z - \tilde{b}_0(t)$$

The Stieltjes function $\tilde{F}(z;t)$ for these polynomials is defined as

$$\tilde{F}(z;t) = \sum_{n=0}^{\infty} \tilde{c}_n(t)z^{-n-1} = \sum_{n=0}^{\infty} i^n c_n(it)z^{-n-1} = -iF(z/i,it)$$

Assume now that the function $\tilde{c}_0(t)$ takes real values on the real axis t and is periodic with some real period $\tilde{c}_0(t+T) = \tilde{c}_0(t)$. Then we can present $\tilde{c}_0(t)$ (at least formally) in terms of the Fourier series

$$\tilde{c}_0(t) = \sum_{k=-\infty}^{\infty} \mu_k \exp(2\pi ikt/T) \tag{1.36}$$

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In this case the measure for the modified polynomials $\tilde{P}_n(z;t)$ is purely discrete and is located on the uniform grid

$$z_k = 2\pi i k/T, \quad k = 0, \pm 1, \pm 2, \ldots \qquad (1.37)$$

on the imaginary axis. Hence, for real and periodic functions $\tilde{c}_0(t)$ we obtain orthogonal polynomials $\tilde{P}_n(z;t)$ with the measure located on the imaginary axis.

Returning to the initial orthogonal polynomials $P_n(z;t)$ we easily obtain

$$F(z;t) = \sum_{k=-\infty}^{\infty} \frac{\mu_k}{z - 2\pi k/T} \exp(2\pi kt/T) \qquad (1.38)$$

We see that in this case the Stieltjes function $F(z;t)$ corresponds to a purely discrete measure located on the *real* axis at points

$$z_k = 2\pi k/T, \ k = 0, \pm 1, \pm 2, \ldots \qquad (1.39)$$

with the corresponding discrete masses

$$M_k(t) = \mu_k \, \exp(2\pi kt/T) \qquad (1.40)$$

The measure is well defined if conditions for the moments

$$c_j(t) = \sum_{k=-\infty}^{\infty} M_k(t) z_k^j = \sum_{k=-\infty}^{\infty} \mu_k \, \exp(2\pi kt/T) z_k^j < \infty \qquad (1.41)$$

hold for all nonnegative integers j (at lest for some values of t).

The trick with passing to the purely imaginary time $t \to it$ will be especially useful if the Fourier series for the modified function $\tilde{c}_0(t)$ is known. Then we can restore information on the initial measure using formula (1.38).

2.   Elliptic functions.

In this section we recall basic properties of the Weierstrass elliptic functions which will be needed in further analysis [2], [26]. The Weierstrass function $\wp(z; g_2, g_3)$ depends on the argument z and two parameters g_2, g_3 (the so-called invariants). It satisfies the differential equation

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_3)(\wp(z) - e_3)$$

where the parameters e_i satisfy the restriction $e_1 + e_2 + e_3 = 0$. The function $\wp(z)$ is double-periodic:

$$\wp(z + 2\omega) = \wp(z + 2\omega') = \wp(z),$$

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where the periods $2\omega, 2\omega'$ are assumed to satisfy the condition $Im(\omega'/\omega) > 0$. If g_2, g_3 are known then the periods $2\omega, 2\omega'$ can be calculated by a standard procedure in terms of the elliptic integrals of the first kind [2].

It is convenient to introduce the notation [2]

$$\omega_1 = \omega, \ \omega_3 = \omega', \ \omega_2 = -\omega - \omega'$$

There is a relation between e_i and ω_i:

$$\wp(\omega_k) = e_k, \quad k = 1, 2, 3 \tag{2.1}$$

The Weierstrass zeta function $\zeta(z)$ is an odd function $\zeta(-z) = -\zeta(z)$ defined as

$$\zeta'(z) = -\wp(z)$$

The function $\zeta(z)$ has simple poles at the points $2m\omega + 2m'\omega'$, where m, m' are arbitrary integers. In contrast to $\wp(z)$, the zeta function $\zeta(z)$ is quasiperiodic:

$$\zeta(z + 2\omega_k) = \zeta(z) + 2\eta_k, \quad k = 1, 2, 3 \tag{2.2}$$

where

$$\eta_k = \zeta(\omega_k)$$

There are useful relations for η_k:

$$\eta_1 + \eta_2 + \eta_3 = 0 \tag{2.3}$$

and

$$\eta_2\omega_1 - \eta_1\omega_2 = i\pi/2$$

(there are two similar relations which are obtained from the last relation by a cyclic permutation of 1,2,3).

The Weierstrass sigma function $\sigma(z)$ is an odd function $\sigma(-z) = -\sigma(z)$ defined as

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z) \tag{2.4}$$

It has simple zeroes at the points $2m\omega + 2m'\omega'$. The sigma function has quasi-periodic property

$$\sigma(z + 2\omega_k) = -\exp(2\eta_k(z + \omega_k))\,\sigma(z) \tag{2.5}$$

There is a simple formula connecting $\wp(z)$ and $\sigma(z)$ [2]:

$$\frac{\sigma(u + v)\sigma(u - v)}{\sigma^2(u)\sigma^2(v)} = \wp(v) - \wp(u) \tag{2.6}$$

Apart from the function $\sigma(z)$ one can define functions $\sigma_k(z), k = 1, 2, 3$ by the formulas [2]

$$\sigma_\alpha(z) = \frac{\sigma(z + \omega_\alpha)}{\sigma(\omega_\alpha)} \exp(-z\eta_\alpha), \quad \alpha = 1, 2, 3 \tag{2.7}$$

The functions $\sigma_\alpha(z)$ are convenient when passing from the Weierstrass to the Jacobi elliptic functions. Indeed, we have [26]

$$\operatorname{sn}(u; k) = (e_1 - e_3)^{1/2} \frac{\sigma(z)}{\sigma_3(z)}, \quad \operatorname{cn}(u; k) = \frac{\sigma_1(z)}{\sigma_3(z)}, \quad \operatorname{dn}(u; k) = \frac{\sigma_2(z)}{\sigma_3(z)} \tag{2.8}$$

where

$$u = (e_1 - e_3)^{1/2} z, \quad k^2 = \frac{e_2 - e_3}{e_1 - e_3} \tag{2.9}$$

The parameter k is called the elliptic modulus. The parameter

$$k' = (1 - k^2)^{1/2} = \sqrt{\frac{e_1 - e_2}{e_1 - e_3}}$$

is called the complementary modulus [26]. The values

$$K = \sqrt{e_1 - e_3}\, \omega_1, \quad K' = i\sqrt{e_1 - e_3}\, \omega_3$$

are complete elliptic integrals of the first kind [26].

We need also expressions of the Weierstrass functions for the value $z = \omega_1/2$:

$$\wp(\omega_1/2) = \wp(3\omega_1/2) = e_1 + (e_1 - e_3)k', \quad \wp(\omega_3 + \omega_1/2) = \frac{e_3 k' + e_2}{1 + k'},$$
$$\wp'(\omega_1/2) = -2k'(1 + k')(e_1 - e_3)^{3/2}, \quad \wp''(\omega_1/2) = 4(e_1 - e_3)\left(2(e_1 - e_2) + 3e_1 k'\right) \tag{2.10}$$

$$2\zeta(\omega_1/2) = \eta_1 - \frac{1}{2}\frac{\wp''(\omega_1/2)}{\wp'(\omega_1/2)} =$$
$$\eta_1 + \sqrt{e_1 - e_3}(k' + 1),$$
$$\zeta(\omega_3 + \omega_1/2) = \eta_3 + \eta_1/2 + \sqrt{e_1 - e_3}(1 - k')/2. \tag{2.11}$$

Note that the choice of an appropriate sign in front of square roots in formulas (2.10) and (2.11) is not trivial problem and depends on location of the parameters e_1, e_2, e_3 in the complex domain. However in our further analysis we will use the "canonical" choice of these parameters: they are real and ordered as $e_3 < e_2 < e_1$. Then all the square roots are assumed in the arithmetic meaning.

Apart from the Weierstrass zeta function $\zeta(z; g_2, g_3)$ sometimes the Jacobi Zeta function $Z(z; k)$ is more convenient. The relation between these function is [26]

$$\zeta(z) = \frac{z\eta_1}{\omega_1} + \sqrt{e_1 - e_3}\left\{Z(u, k) + \frac{\operatorname{cn}(u, k)\operatorname{dn}(u, k)}{\operatorname{sn}(u, k)}\right\}, \tag{2.12}$$

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where the same relations (2.9) are assumed.

The Jacobi Zeta function is purely periodic with respect to the period $2K$:

$$Z(u + 2K) = Z(u)$$

and quasi-periodic with respect to the period $2iK'$:

$$Z(u + 2iK') = Z(u) - \frac{i\pi}{K}$$

It possesses a remarkable "addition theorem" [26]

$$Z(u + v) = Z(u) + Z(v) - k^2 \text{sn}(u)\text{sn}(v)\text{sn}(u + v) \tag{2.13}$$

Note the useful relations

$$Z(u + K) = Z(u) - k^2 \frac{\text{sn}(u)\text{cn}(u)}{\text{dn}(u)} \tag{2.14}$$

$$Z(u + iK') = Z(u) + \frac{\text{dn}(u)\text{cn}(u)}{\text{sn}(u)} - \frac{i\pi}{2K} \tag{2.15}$$

$$Z(u + K + iK') = Z(u) - \frac{\text{sn}(u)\text{dn}(u)}{\text{cn}(u)} - \frac{i\pi}{2K}. \tag{2.16}$$

The Jacobi Zeta function can be expressed in terms of the incomplete elliptic integral $E(u)$ of the second kind [26]

$$Z(u) = E(u) - u\frac{E}{K} \tag{2.17}$$

where

$$E(u) = \int_0^u \text{dn}^2(t)dt$$

and $E = E(K)$. The following formulas allow to express the functions $\zeta(z + \omega_i)$, $i = 1, 2, 3$ in terms of the Jacobi Zeta function:

$$\zeta(z + \omega_1) = \eta_1 + \frac{z\eta_1}{\omega_1} + \sqrt{e_1 - e_3}\left\{Z(u, k) - \frac{\text{sn}(u, k)\text{dn}(u, k)}{\text{cn}(u, k)}\right\}, \tag{2.18}$$

$$\zeta(z + \omega_3) = \eta_3 + \frac{z\eta_1}{\omega_1} + \sqrt{e_1 - e_3}Z(u, k), \tag{2.19}$$

$$\zeta(z + \omega_1 + \omega_3) = -\eta_2 + \frac{z\eta_1}{\omega_1} + \sqrt{e_1 - e_3}\left\{Z(u, k) - k^2\frac{\text{sn}(u, k)\text{cn}(u, k)}{\text{dn}(u, k)}\right\}, \tag{2.20}$$

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3. Toda chain solution and the corresponding orthogonal polynomials

We present here an explicit solution of the restricted Toda chain

Lemma 1 *Put*
$$u_n(t) = w^2 n^2 \left(\wp(w(t+\beta)) - \wp(nw(t+\beta)+q) \right) \tag{3.1}$$

and

$$b_n(t) = \mu_1 + w(n+1)\zeta(w(n+1)(t+\beta)+q) - wn\zeta(wn(t+\beta)+q) - (2n+1)w\zeta(w(t+\beta)), \tag{3.2}$$

where w, β, q, μ_1 are arbitrary complex parameters. We will also assume that ω_1, ω_3 are arbitrary independent periods corresponding to the arbitrary parameters e_1, e_2, e_3 with the only condition $e_1 + e_2 + e_3 = 0$.

Then $u_n(t), b_n(t)$ satisfy the restricted Toda chain equations (1.1)

Proof. In order to verify the first equation in (1.1) we present $u_n(t)$ in an equivalent form

$$u_n(t) = w^2 n^2 \frac{\sigma((n+1)w(t+\beta)+q)\sigma((n-1)w(t+\beta)+q)}{\sigma^2(nw(t+\beta)+q)\sigma^2(w(t+\beta))} \tag{3.3}$$

using formula (2.6). Then by (2.4) the expression $\dot{u}_n/u_n$ can be presented as a sum of the Weierstrass zeta functions and we arrive at the first equation of (1.1). The second equation (1.1) is satisfied by the formula $\wp(z) = -\zeta'(z)$.

It is directly verified that the recurrence coefficients $u_n(t), b_n(t)$ are double-periodic with the periods $2\omega/w, 2\omega'/w$:

$$u_n(t+2\omega/w) = u_n(t+2\omega'/w) = u_n(t), \quad b_n(t+2\omega/w) = b_n(t+2\omega'/w) = b_n(t)$$

(periodicity property for $u_n(t)$ is obvious and periodicity for the coefficients $b_n(t)$ follows from (2.2) and (2.3)). Thus both $u_n(t)$ and $b_n(t)$ are elliptic functions in the argument t.

Using formulas (1.19) and (2.4) we can restore the function $c_0(t)$. It is easy to verify that

$$c_0(t) = \frac{\sigma(w(t+\beta)+q)}{\sigma(q)\,\sigma(w(t+\beta))} \exp(\mu_1 t + \mu_0), \tag{3.4}$$

where μ_0 is an arbitrary constant.

We thus obtained some new family of orthogonal polynomials $P_n(z;t)$ which can be defined through given recurrence coefficients $u_n(t), b_n(t)$. The Stieltjes function $F(z,t)$ (and hence, in principle) the orthogonality measure for these polynomials can also be found explicitly from formula (1.25) because the function $c_0(t)$ is given explicitly by (3.4).

The obtained orthogonal polynomials $P_n(z;t)$ contain several free parameters (say w, q, β, μ_1, t and elliptic parameters g_2, g_3). We would like to investigate some simple special choice of these parameters when our polynomials are a generalization of already known families. Note that the parameter β is inessential: it describes a shift of the argument $t \to t + \beta$. Nevertheless, we will keep this parameter for convenience, assuming that the argument t takes real or pure imaginary values. We will assume also that $q \neq 0$. Indeed, the case $q = 0$ corresponds to some degeneration: $b_0(t) = const$, $u_1(t) = 0$ and $c_0(t)$ becomes a pure exponential function $c_0 = \exp(\mu t)$ in this limit.

In what follows we put $q = \omega_j$, $\beta = \omega_k/w$, where k, j are arbitrary noncoinciding integers from the set $1, 2, 3$. We denote also $\omega_l = -\omega_j - \omega_k$. Then using (quasi)periodicity properties of the Weierstrass functions $\wp(z), \zeta(z)$ we find the expression for $u_n(t)$:

$$u_{2n}(t) = 4w^2 n^2(\wp(wt + \omega_k) - \wp(2wnt + \omega_j))$$
$$u_{2n+1}(t) = w^2(2n+1)^2(\wp(wt + \omega_k) - \wp(w(2n+1)t + \omega_l)) \tag{3.5}$$

and for the coefficients $b_n(t)$:

$$b_{2n}(t) = \mu_1 + w\left\{(2n+1)\zeta((2n+1)wt - \omega_l) - 2n\zeta(2nwt + \omega_j) - (4n+1)\zeta(wt + \omega_k) + 2n\eta_k\right\},$$
$$b_{2n+1}(t) = \mu_1 + w\left\{2(n+1)\zeta((2(n+1)wt + \omega_j) - (2n+1)\zeta((2n+1)wt - \omega_l) -\right.$$
$$\left.(4n+3)\zeta(wt + \omega_k) + (6n+4)\eta_k\right\} \tag{3.6}$$

We will assume also that $\mu_0 = 0$ and

$$\mu_1 = -w\eta_j, \tag{3.7}$$

Indeed, the parameter μ_0 is inessential and can be chosen arbitrary whereas the parameter μ_1 leads only to a trivial shift of the recurrence coefficient b_n, hence we can put μ_1 to a prescribed value without loss of generality.

In order to find the orthogonality measure for the obtained polynomials $P_n(z;t)$ we will assume that the parameter w is real. Among all 6 possible choices $q = \omega_j$, $\beta = \omega_k/w$ of the parameters q, β we consider only the two cases:

(i) if $\beta = \omega_1/w$, $q = \omega_2$ then form (2.7), (3.7) and (2.8) we find that

$$c_0(t) = C_1/\text{cn}(w\sqrt{e_1 - e_3}\,t; k) \tag{3.8}$$

where

$$C_1 = -\frac{\sigma(\omega_3)}{\sigma(\omega_1)\sigma(\omega_2)}\, e^{-\omega_1\eta_2}$$

(ii) if $\beta = \omega_1/w$, $q = \omega_3$ then quite analogously we find

$$c_0(t) = C_2 \frac{\mathrm{dn}(w\sqrt{e_1 - e_3}t; k)}{\mathrm{cn}(w\sqrt{e_1 - e_3}t; k)} = C_2 \, \mathrm{dc}(w\sqrt{e_1 - e_3}t; k) \qquad (3.9)$$

where

$$C_2 = -\frac{\sigma(\omega_2)}{\sigma(\omega_1)\sigma(\omega_3)} \, e^{-\omega_1 \eta_3}$$

Note that the constant factors C_1, C_2 are in fact inessential (the orthogonal polynomials $P_n(z;t)$ as well as recurrence coefficients $b_n(t), u_n(t)$ do not depend on these constants) and we can put $C_2 = C_1 = 1$. We thus have that $c_0(t) = 1/\mathrm{cn}(wt\sqrt{e_1 - e_3}; k)$ for the case (i) and $c_0(t) = \mathrm{dc}(w\sqrt{e_1 - e_3}t; k))$ for the case (ii).

We will restrict ourselves with the case when all e_i are real and distinct, say $e_1 > e_2 > e_3$ (this is the usual convention [2]). Then it is well known [2] that for the real values of z the functions $\wp(z), \zeta(z), \sigma(z)$ take real values. For purely imaginary values of z the function $\wp(z)$ takes real values whereas functions $\zeta(z), \sigma(z)$ take purely imaginary values. The period $2\omega_1$ is real and the period $2\omega_3$ is purely imaginary. This means that the fundamental parallelogram of the elliptic functions in this case is a rectangle [2]. We have also that both k and k' are real parameters taking values in the "canonical" interval $0 < k, k' < 1$. Hence all values of Jacobi elliptic functions $\mathrm{sn}(x), \mathrm{cn}(x), \mathrm{dn}(x)$ are real for real x. The functions $\mathrm{cn}(x), \mathrm{dn}(x)$ take also real values for purely imaginary values of x.

Now we are ready to calculate the orthogonality measure for the cases (i) and (ii).

4. The orthogonality measure and recurrence coefficients for the case (i)

According to considerations of the first section, introduce the new function $\tilde{c}_0(t) = c_0(it)$.

For the first case (i) we have

$$\tilde{c}_0(t) = 1/\mathrm{cn}(iw\sqrt{e_1 - e_3}t; k) = \mathrm{cn}(w\sqrt{e_1 - e_3}t; k') \qquad (4.1)$$

The function $\mathrm{cn}(w\sqrt{e_1 - e_3}t; k')$ is real on the real axis and periodic with the period $T = \frac{4K'}{w\sqrt{e_1 - e_3}}$. The Fourier series for this function is well known [26]

$$\mathrm{cn}(w\sqrt{e_1 - e_3}t; k') = \frac{\pi}{k'K'} \sum_{n=-\infty}^{\infty} \frac{1}{v^{n-1/2} + v^{1/2-n}} \exp(\pi i(n - 1/2)w\sqrt{e_1 - e_3}t/K'), \qquad (4.2)$$

where

$$v = \exp(-\pi K/K')$$

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Hence the polynomials $P_n(z;t)$ have a purely discrete orthogonality measure located at the points

$$x_n = \frac{2\pi}{T}(2n-1) = \frac{\pi w\sqrt{e_1-e_3}}{2K'}(2n-1), \quad n = 0, \pm 1, \pm 2, \ldots \tag{4.3}$$

with the corresponding concentrated masses

$$M_n(t) = \frac{\pi}{k'K'}\frac{\exp(\pi wt(n-1/2)\sqrt{e_1-e_3}/K')}{v^{n-1/2} + v^{1/2-n}} \tag{4.4}$$

Thus orthogonality relation for the polynomials $P_n(z;t)$ looks as follows

$$\sum_{s=-\infty}^{\infty} M_s(t)P_n(x_s;t)P_m(x_s;t) = h_n(t)\,\delta_{nm} \tag{4.5}$$

It is interesting to determine conditions under which the measure is well defined, i.e. that all the moments are finite

$$\tilde{c}_j(t) = \sum_{s=-\infty}^{\infty} M_s(t)x_s^j < \infty, \quad j = 0, 1, 2, \ldots \tag{4.6}$$

From the explicit expression (4.4) it is easily seen that condition (4.6) will hold provided

$$-\frac{K}{w\sqrt{e_1-e_3}} < t < \frac{K}{w\sqrt{e_1-e_3}} \tag{4.7}$$

When the parameter t belongs to this interval all the moments are well defined. If $t \to \pm\frac{K}{w\sqrt{e_1-e_3}}$ then $c_0(t) \to \infty$ as is easily seen from (4.1). Hence when t approaches the endpoints of the interval (4.7), the moments $c_n(t)$ tend to infinity and the measure becomes not well defined. From (4.4) it is clear that for all values of t from the admissible interval (4.7) the concentrated masses are positive $M_s(t) > 0$, $s = 0, \pm 1, \pm 2, \ldots$. This means that we indeed deal with a positively defined purely discrete measure on the whole real axis.

Moreover it is easy verified that

$$\sum_{s=-\infty}^{\infty} M_s(t)x_s^j = c_j(t) = \frac{d^j}{dt^j}c_0(t) \tag{4.8}$$

where $c_0(t) = 1/\mathrm{cn}(wt\sqrt{e_1-e_3};k)$. Indeed, formula (4.8) follows directly from the Fourier series (4.2) by j-fold differentiation with respect to t. Formula (4.8) shows that the obtained measure is "true", i.e. it gives the prescribed moments $c_j(t)$ for all $j = 0, 1, \ldots$.

Consider the recurrence coefficients $b_n(t), u_n(t)$ for the orthogonal polynomials $P_n(z;t)$ corresponding to the function $\tilde{c}_0(t)$ defined by (4.1).

It is clear that both $b_n(t)$ and $u_n(t)$ are real for all t from the admissible interval (4.7). Indeed, the Hankel determinants $D_n(t)$ are real because these are constructed from the matrix with real entries

$$a_{ij} = \frac{d^{i+j}c_0(t)}{dt^{i+j}}, \quad i, j = 0, 1, \ldots, n-1$$

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Hence the normalization coefficients $h_n(t)$ are real as well. The same is true for coefficients $b_n(t), u_n(t)$ obtained from $h_n(t)$ by formulas

$$b_n(t) = \frac{\dot{h}_n}{h_n}, \quad u_n = h_n/h_{n-1}$$

Explicitly we have for the recurrence coefficients $b_n(t)$ (see (3.6))

$$b_{2n}(t) = w\left\{(2n+1)\zeta((2n+1)wt - \omega_3) - 2n\zeta(2nwt + \omega_2) - (4n+1)\zeta(wt + \omega_1) + 2n\eta_1 - \eta_2\right\},$$

$$b_{2n+1}(t) = w\left\{2(n+1)\zeta((2(n+1)wt + \omega_2) - (2n+1)\zeta((2n+1)wt - \omega_3) - \right.$$

$$(4n+3)\zeta(wt + \omega_1) + (6n+4)\eta_1 - \eta_2\right\} \tag{4.9}$$

or, in terms of the elliptic Jacobi functions

$$b_{2n}(t) = w\sqrt{e_1 - e_3}\left\{(2n+1)Z((2n+1)u) - 2nZ(2nu) - (4n+1)Z(u) - \right.$$

$$2nk^2\frac{\text{cn}(2nu)\text{sn}(2nu)}{\text{dn}(2nu)} + (4n+1)\frac{\text{sn}(u)\text{dn}(u)}{\text{cn}(u)}\right\} = \tag{4.10}$$

$$w\sqrt{e_1 - e_3}\left\{(2n+1)Z((2n+1)u) - 2nZ(2nu + K) - (4n+1)\left(Z(u + K + iK') + \frac{i\pi}{2K}\right)\right\}$$

$$b_{2n+1}(t) = w\sqrt{e_1 - e_3}\left\{2(n+1)Z(2(n+1)u) - (2n+1)Z((2n+1)u) - (4n+3)Z(u) - \right.$$

$$2(n+1)k^2\frac{\text{cn}(2(n+1)u)\text{sn}(2(n+1)u)}{\text{dn}(2(n+1)u)} + (4n+3)\frac{\text{sn}(u)\text{dn}(u)}{\text{cn}(u)}\right\} = \tag{4.11}$$

$$w\sqrt{e_1 - e_3}\left\{2(n+1)Z(2(n+1)u + K) - (2n+1)Z((2n+1)u) - (4n+3)\left(Z(u) + \frac{i\pi}{2K}\right)\right\}$$

where

$$u = w\,t\,\sqrt{e_1 - e_3}$$

For the recurrence coefficients $u_n(t)$ we have expressions in terms of the elliptic Jacobi functions

$$u_{2n}(t) = 4n^2 w^2(e_1 - e_2)\left(\frac{1}{\text{cn}^2(u)} + k^2\frac{\text{sn}^2(2nu)}{\text{dn}^2(2nu)}\right)$$

$$u_{2n+1}(t) = (2n+1)^2 w^2(e_1 - e_3)\left(k'^2\frac{\text{sn}^2(u)}{\text{cn}^2(u)} + \text{dn}^2((2n+1)u)\right) \tag{4.12}$$

It is seen from (4.12) that for all t from the admissible interval the recurrence coefficients $u_n(t)$ are bounded and strictly positive $u_n(t) > 0$. As is well known from general theory of orthogonal polynomials [8] the property $u_n > 0$ for all $n > 0$ (together with reality of

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the coefficients b_n) guarantees existence of a positive measure on the real axis. We already constructed this measure explicitly (4.5).

The remaining question is about uniqueness of the moment problem for the case (i). Indeed, we have constructed explicitly the orthogonality measure (4.5) corresponding to the moments

$$c_n(t) = \frac{d^n}{dt^n} \left\{ \frac{1}{\text{cn}(wt\sqrt{e_1 - e_3}; k)} \right\}, \quad n = 0, 1, 2, \ldots, \tag{4.13}$$

where t is assumed to belong to the admissible interval (4.7).

But in principle, it is possible that this measure is not unique. Such situation is known as indeterminate moment problem [1], [20]. In more details this means the following. Assume that real moments c_n are given and all corresponding the Hankel determinants $D_n > 0$ are positive. This condition is equivalent to positivity of the recurrence coefficients $u_n > 0$ for $n = 1, 2, \ldots$ and in turn, it guarantees existence of a positive orthogonality measure on the real line $-\infty < x < \infty$ (so-called the Hamburger moment problem [20], [9]). If this measure is unique (up to a normalization condition) then the Hamburger moment problem is called the *determinate*. If there exist at least two different orthogonality measures then the Hamburger moment problem is called indeterminate. In case of the indeterminate Hamburger problem there exists infinitely many different measures (see [1], [20] for details).

Finding criteria for determinacy of the Hamburger moment problem is a nontrivial problem [9]. However, there is a simple *sufficient* condition proposed by Carleman [20]: if

$$\sum_{n=1}^{\infty} u_n^{-1/2} = \infty \tag{4.14}$$

then the Hamburger problem is determinate. Of course, in (4.14) the arithmetic value of the square root $\sqrt{u_n}$ is assumed.

We now show that the Hamburger problem for the moment problem (4.13) is determinate.

From (4.12) it follows that for all n and for fixed t from the admissible interval we have the inequalities

$$0 < u_{2n}(t) < A(t)$$

where

$$A(t) = \frac{1}{\text{cn}^2(u)} + \frac{k^2}{k'^2} = \frac{\text{dn}^2(u)}{k'^2 \text{cn}^2(u)}$$

is a fixed positive parameter (depending on t but not on n). Hence we have

$$\sum_{n=1}^{\infty} u_{2n}^{-1/2} = \infty$$

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diverges. From the similar considerations it follows that

$$\sum_{n=1}^{\infty} u_{2n+1}^{-1/2} = \infty$$

Hence the Carleman condition (4.14) holds and we indeed have the determinate moment problem. This means that the discrete measure (4.5) is the only providing orthogonality of the polynomials $P_n(z;t)$ on the real axis.

When t approaches the endpoints of the admissible interval (4.7) the recurrence coefficients $b_n(t), u_n(t)$ tend to infinity. This "explosion" of the recurrence coefficients explains corresponding "explosion" of the orthogonality measure when t tends to he endpoints of the admissible interval.

Put now $t = 0$ (the middlepoint of the admissible interval) . Then it is seen (due to property $\eta_1 + \eta_2 + \eta_3 = 0$) that the recurrence coefficient $b_n(0)$ vanishes

$$b_n(0) = 0, \quad n = 0, 1, 2, \ldots \tag{4.15}$$

and we arrive at a class of so-called symmetric orthogonal polynomials with the recurrence relation [8]

$$P_{n+1}(z) + u_n(0)P_{n-1}(z) = zP_n(z) \tag{4.16}$$

For the recurrence coefficients $u_n(0)$ we have

$$u_{2n}(0) = 4w^2 n^2 (e_1 - e_2), \quad u_{2n+1}(0) = w^2 (2n+1)^2 (e_1 - e_3). \tag{4.17}$$

These recurrence coefficients correspond to the orthogonal polynomials introduced by Carlitz [7]. In turn, orthogonality relation for these OP follows from the remarkable result by Stieltjes on presenting of the Laplace transform of the Jacobi elliptic functions in terms of continued fraction (for modern treatment of this and related examples see e.g. [16]. Today the orthogonal polynomials introduced by Carlitz are called the Stieltjes-Carlitz orthogonal polynomials related with elliptic functions [8]. For further details concerning these polynomials see [15].

The orthogonality measure for polynomials corresponding to the Stieltjes-Carlitz case (4.17) is obtained from our measure by putting $t = 0$. We thus have the orthogonality relation

$$\sum_{s=-\infty}^{\infty} M_s(0)P_n(x_s;0)P_m(x_s;0) = h_n(0)\,\delta_{nm} \tag{4.18}$$

where the support of the measure is the same, i.e. the points x_s have the same expression (4.3) and the concentrated masses are

$$M_s(0) = \frac{\pi}{k'K'} \frac{1}{v^{s-1/2} + v^{1/2-s}} \tag{4.19}$$

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This measure was discovered by Stieltjes (Carlitz showed that this measure provides orthog-onality of the corresponding polynomials $P_n(z)$). For the Stieltjes-Carlitz polynomials the Hamburger moment problem is obviously determined because $t = 0$ belongs to the admissible interval.

We see that the Stieltjes-Carlitz polynomials appear naturally as a very special case of the "elliptic Toda polynomials" corresponding to "zero time" condition $t = 0$.

We can finally summarize all these results as the

Theorem 1 *Assume that $e_3 < e_2 < e_1$ are arbitrary real parameters with the condition $e_1 + e_2 + e_3 = 0$. Assume that the recurrence coefficients are given by formulas (4.9), (4.12) with arbitrary positive parameter w. Assume also that the parameter t belongs to the admissible interval (4.7). Then the corresponding orthogonal polynomials $P_n(x;t)$ are orthogonal on the uniform grid (4.3) on the real axis with the concentrated masses given by (4.4). The corresponding moments $c_n(t)$ are given by (4.13). The moment problem is determinate.*

5. The orthogonality measure and recurrence coefficients for the case (ii)

Consider now the case (ii). We have analogously

$$\tilde{c}_0(t) = \mathrm{dc}(iw\sqrt{e_1 - e_3}t; k) = \mathrm{dn}(w\sqrt{e_1 - e_3}t; k') \tag{5.1}$$

The Fourier series is well known [26]

$$\mathrm{dn}(w\sqrt{e_1 - e_3}t; k') = \frac{\pi}{K'} \sum_{n=-\infty}^{\infty} \frac{1}{v^n + v^{-n}} \exp(\pi inwt\sqrt{e_1 - e_3}/K') \tag{5.2}$$

with the same expression for h as for the case (i). From considerations of the first section we see that corresponding orthogonal polynomials $P_n(z;t)$ have purely discrete measure located at the points

$$x_n = \frac{\pi w\sqrt{e_1 - e_3}n}{K'}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{5.3}$$

Corresponding concentrated masses are

$$M_n(t) = \frac{2\pi}{K'(v^n + v^{-n})} \exp(\pi wn\sqrt{e_1 - e_3}t/K') \tag{5.4}$$

The admissible interval for t is the same as for the case (i):

$$-\frac{K}{w\sqrt{e_1 - e_3}} < t < \frac{K}{w\sqrt{e_1 - e_3}}$$

As for the case (i) it is easily verified that inside the admissible interval the recurrence coeffi-
cients $b_n(t), u_n(t)$ are real and $u_n > 0$ which guarantees positivity of the measure: $M_n(t) > 0$
for all $n = 0, \pm 1, \pm 2$ and for all t from the admissible interval.

The coefficients $u_n(t)$ has the expression

$$u_{2n}(t) = 4w^2 n^2 (e_1 - e_3) \left(k'^2 \frac{\operatorname{sn}^2(u)}{\operatorname{cn}^2(u)} + \operatorname{dn}^2(2nu) \right)$$

$$u_{2n+1}(t) = w^2 (2n+1)^2 (e_1 - e_2) \left(\frac{1}{\operatorname{cn}^2(u)} + k^2 \frac{\operatorname{sn}^2((2n+1)u)}{\operatorname{dn}^2((2n+1)u)} \right) \qquad (5.5)$$

The recurrence coefficients $b_n(t)$ are expressed as

$$b_{2n}(t) = w\sqrt{e_1 - e_3} \; \{(2n+1)Z((2n+1)u + K) -$$
$$2nZ(2nu) - (4n+1)(Z(u + K + iK') - i\pi/(2K))\}$$

$$(5.6)$$

$$b_{2n+1}(t) = w\sqrt{e_1 - e_3} \; \{2(n+1)Z(2(n+1)u) -$$
$$(2n+1)Z((2n+1)u + K) - (4n+3)(Z(u + K + iK') - i\pi/(2K))\}$$

From (5.5) it is clear that the coefficients $u_n(t)$ are strictly positive $u_n(t) > 0$ for any fixed
value of the parameter t from the admissible interval.

From the same considerations it follows that the Hamburger moment problem for the mo-
ments

$$c_n(t) = \frac{d^n}{dt^n} \left\{ \frac{\operatorname{dn}(wt\sqrt{e_1 - e_3}t; k)}{\operatorname{cn}(wt\sqrt{e_1 - e_3}t; k)} \right\}, \quad n = 0, 1, 2, \ldots$$

is determinate for any value of the parameter t from the admissible interval (4.7).

When $t = 0$ then again, as in the case (i) the diagonal recurrence coefficients are zero
$b_n(0) = 0$ and

$$u_{2n} = 4w^2 n^2 (e_1 - e_3), \quad u_{2n+1} = w^2 (2n+1)^2 (e_1 - e_2) \qquad (5.7)$$

Note that the recurrence coefficients (5.7) are obtained from the corresponding coefficients
(4.17) of the case (i) by a simple transposition $e_2 \leftrightarrow e_3$. These recurrence coefficients (5.7)
correspond to the second class of the Stieltjes-Carlitz orthogonal polynomials [7], [15] arising
from the Laplace transformation of the elliptic function $\operatorname{dn}(t; k')$.

We see, that just as in the case (i), the special choice of the parameter $t = 0$ leads to already
known Stieltjes-Carlitz orthogonal polynomials. For arbitrary values of the parameter t from
the admissible interval we obtain new orthogonal polynomials with explicitly known recurrence
coefficients and positive discrete measure on a whole real axis.

We considered only two possible choices of the parameters β, q ($\beta = \omega_1/w$, $q = \omega_2$ and
$\beta = \omega_1/w$, $q = \omega_3$) because for these two choices we get polynomials $P_n(z; t)$ having positive

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orthogonality measure on the real axis. It seems that all other values of the parameters β, q (for real values w) do not lead to polynomials with a positive measure. Nevertheless, as we will see in the next sections, in the degenerate cases of elliptic functions there are more possibilities for these parameters when the measure appears to be a positive.

6. Special choices of the parameter t

The expressions (3.5) and (3.6) show that for generic value of t the recurrence coefficients $b_n(t), u_n(t)$ are transcendent functions in n. We already saw that for special choice $t = 0$ we obtain $b_n(0) = 0$ and $u_{2n}(0)$ as well $u_{2n+1}(0)$ are quadratic polynomials in n.

Put now
$$t = \frac{M\,\omega_1}{N\,w}, \tag{6.1}$$
where $M < N$ are mutually prime positive integers. Present the number n in the form
$$n = Nr + s, \quad r = 0, 1, \ldots, \quad s = 0, 1, \ldots, N - 1$$

Then for fixed $s = 0, 1, \ldots, N - 1$ the recurrence coefficients $b_{2n}(t)$, $b_{2n+1}(t)$ will be linear function in r and the coefficients $u_{2n}(t)$, $u_{2n+1}(t)$ will be quadratic polynomials in r .

Indeed, from formulas (3.5), (3.6), using periodicity of the Weierstrass functions, we get
$$u_{2Nj+2s} = 4w^2(Nr + s)^2 \left(\epsilon_0 - \epsilon_1(s)\right), \tag{6.2}$$
$$u_{2Nr+2s+1} = w^2(2Nr + 2s + 1)^2 \left(\epsilon_0 - \epsilon_2(s)\right),$$

where
$$\epsilon_0 = \wp((M + N)\omega_1/N), \quad \epsilon_1(s) = \wp(\omega_j + 2sM\omega_1/N), \quad \epsilon_2(s) = \wp(\omega_l + (2s + 1)M\omega_1/N)$$

In these formulas $j = 2, l = 3$ for the case (i) and $j = 3, l = 2$ for the case (ii).

Analogously, for the coefficients $b_{2n}(t)$ and $b_{2n+1}(t)$ we obtain
$$b_{2Nr+2s} = w(-\eta_j + 2\eta_1(Mr + Nr + s)) + w(\kappa_1(s)(2Nr + 2s + 1)) -$$
$$2\kappa_2(s)(Nr + s) - \kappa_0(4(Nr + s) + 1)) \tag{6.3}$$

$$b_{2Nr+2s+1} = w(-\eta_j + 2\eta_1(Mr + 3Nr + 3s + 2)) + w(\kappa_3(s)(2Nr + 2s + 2)) -$$
$$\kappa_4(s)(2Nr + 2s + 1) - \kappa_0(4(Nr + s) + 3)) \tag{6.4}$$

where
$$\kappa_0 = \zeta((M + N)\omega_1/N), \quad \kappa_1(s) = \zeta(-\omega_l + (2s + 1)M\omega_1/N), \quad \kappa_2(s) = \zeta(\omega_j + 2sM\omega_1/N)$$

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$$\kappa_3(s) = \zeta(\omega_j + 2sM\omega_1/N), \ \kappa_4(s) = \zeta(-\omega_l + sM\omega_1/N)$$

We see that indeed $u_{2Nr+2s}, u_{2Nr+2s+1}$ are quadratic in r and $b_{2Nr+2s}, b_{2Nr+2s+1}$ are linear in n.

The constants $\epsilon_i(s), \kappa_i(s)$ can be found in a less or more "explicit" form only for several values of N, M. We already considered the case $t = 0$ which corresponds to $M = 0$, $N = 1, s = 0$. Another simplest case corresponds to the choice

$$M = 1, N = 2$$

i.e. we choose $t = \frac{\omega_1}{2w}$. In this case the parameter s can take only 2 values $s = 0, 1$.

The zero moment becomes now

$$c_0(t) = 1/\mathrm{cn}(K/2; k) = \sqrt{\frac{1 + k'}{k'}}.$$

Using relations and (2.11) we can calculate the recurrence coefficients $b_n(\omega_1/(2w))$

$$b_{4n}(\omega_1/(2w)) = w\sqrt{e_1 - e_3}(2n(k' + 3) + 1) \tag{6.5}$$

$$b_{4n+1}(\omega_1/(2w)) = w\sqrt{e_1 - e_3}(2n(3k' + 1) + 2k' + 1) \tag{6.6}$$

$$b_{4n+2}(\omega_1/(2w)) = w\sqrt{e_1 - e_3}(2n(3k' + 1) + 4k' + 1) \tag{6.7}$$

$$b_{4n+3}(\omega_1/(2w)) = w\sqrt{e_1 - e_3}(2n(k' + 3) + 2k' + 5) \tag{6.8}$$

Similarly, using relations (2.10) we obtain expressions for the coefficients $u_n(\omega_1/(2w))$;

$$u_{2n+1}(\omega_1/(2w)) = w^2(e_1 - e_3)(2n + 1)^2 \, 2k' \tag{6.9}$$

$$u_{4n}(\omega_1/(2w)) = 16w^2(e_1 - e_3)n^2 k'(1 + k') \tag{6.10}$$

$$u_{4n+2}(\omega_1/(2w)) = 4w^2(e_1 - e_3)(2n + 1)^2(1 + k') \tag{6.11}$$

Note that the combination $w\sqrt{e_1 - e_3}$ plays the role of a scaling parameter, hence we can put $w = 1$, $e_1 - e_3 = 1$ without loss of generality.

We then have

Theorem 2 *Assume that $0 < k' < 1$ is an arbitrary parameter. Let the recurrence coefficients b_n be defined as*

$$
\begin{aligned}
&(k'+3)n/2+1, \quad if \quad n=0 \quad (mod \quad 4); \\
&(3k'+1)n/2+(k'+1)/2, \quad if \quad n=1 \quad (mod \quad 4); \\
&(3k'+1)n/2+k', \quad if \quad n=2 \quad (mod \quad 4); \\
&(k'+3)n/2+(k'+1)/2 \quad if \quad n=3 \quad (mod \quad 4)
\end{aligned}
\tag{6.12}
$$

and the recurrence coefficients u_n be defined as

$$
\begin{aligned}
&k'(k'+1)n^2, \quad if \quad n=0 \quad (mod \quad 4); \\
&2k'n^2, \quad if \quad n=1,3 \quad (mod \quad 4); \\
&(1+k')n^2, \quad if \quad n=2 \quad (mod \quad 4)
\end{aligned}
\tag{6.13}
$$

Then corresponding monic orthogonal polynomials $P_n(x)$ are orthogonal with purely discrete measure on the real line

$$
\sum_{s=-\infty}^{\infty} M_s P_n(x_s) P_m(x_s) = h_n\, \delta_{nm},
\tag{6.14}
$$

where the grid of orthogonality is

$$
x_s = \frac{\pi(s-1/2)}{K'}
$$

and corresponding concentrated masses are

$$
M_s = \frac{\pi}{k'K'} \frac{v^{(s-1/2)/2}}{v^{s-1/2}+v^{1/2-s}}, \quad v = \exp(-\pi K/K')
$$

The normalization constants h_n are

$$
h_n = c_0 u_1 u_2 \ldots u_n
$$

where

$$
c_0 = \sqrt{\frac{1+k'}{k'}}
$$

7. Degenerated cases

Consider degenerated cases of obtained orthogonal polynomials. These degenerated cases arise when two or three of the parameters e_i coincide. Using our choice $e_1 > e_2 > e_3$ we see that there are two possibilities when two of the parameters coincide:

(i) $e_1 = e_2 = a$, $e_3 = -2a$, where a is a positive parameter. In this case the real period $2\omega_1$ tends to infinity, whereas the imaginary period remains finite

$$\omega_3 = \frac{\pi i}{\sqrt{12a}}$$

Without loss of generality we can take $a = 1/3$ (changing of a leads only to scaling of the argument z of corresponding functions). The Weierstrass functions are then reduced to hyperbolic ones, e.g.

$$\wp(z) \rightarrow 1/3 + \frac{1}{\sinh^2(z)}$$

The modular parameter becomes $k = 1$ and the Jacobi elliptic functions become hyperbolic as well: $\operatorname{sn}(z; 1) = \tanh(z)$, $\operatorname{cn}(z; 1) = \operatorname{dn}(z; 1) = 1/\cosh(z)$. The imaginary period in this case is $2\omega_3 = i\pi$.

(ii) $e_2 = e_3 = -a$, $e_1 = 2a$ with some positive parameter a. Then the imaginary period $2\omega_3$ becomes infinity whereas the real period is finite

$$\omega_1 = \frac{\pi}{\sqrt{12a}}$$

Again we can put $a = 1/3$, then the Weierstrass function $\wp(z)$ is degenerated to trigonometric form:

$$\wp(z) \rightarrow -1/3 + \frac{1}{\sin^2(z)}$$

The modular parameter $k = 0$ in this limit and we have $\operatorname{sn}(z; 0) = \sin(z)$, $\operatorname{cn}(z; 0) = \cos(z)$, $\operatorname{dn}(z; 0) = 1$.

Consider first the degenerated cases (i) and (ii) of the elliptic polynomials obtained in the previous section. In the "hyperbolic" limit $k = 1$ the functions $c_0(t)$ becomes the same $c_0(t) = \cosh(t)$. This case is non-interesting because it corresponds to degeneration of the orthogonal polynomials: the Hankel determinants become zero $D_n(t)$ for infinitely many n. In turn, this corresponds to zero recurrence coefficients for even $2n$: $u_{2n}(t) = 0$ as can be seen from explicit formulas for $u_n(t)$ in the hyperbolic limit.

In the trigonometric limit $k = 0$ again both functions $c_0(t)$ coincide: $c_0(t) = 1/\cos(t)$. This function $c_0(t)$ corresponds to some elementary solutions of the restricted Toda chain; corresponding orthogonal polynomials $P_n(z; t)$ coincide with a special case of the Meixner-Pollaczeck polynomials.

Indeed, for this case it is elementary verified that the recurrence coefficients have the expression:

$$u_n(t) = \frac{n^2}{\cos^2 t}, \quad b_n(t) = (2n + 1)\,\tan t \tag{7.1}$$

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On the other hand the monic Meixner-Pollaczeck polynomials [13]

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n e^{in\phi}}{(2\sin\phi)^n} {}_2F_1 \left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} ; 1 - e^{-2i\phi} \right) \tag{7.2}$$

(here $(a)_n = a(a+1)\ldots(a+n-1)$ is the standard shifted factorial) depend on two parameters λ, ϕ and have the recurrence coefficients

$$u_n = \frac{n(n+2\lambda-1)}{4\sin^2\phi}, \quad b_n = -\frac{n+\lambda}{\tan\phi} \tag{7.3}$$

Comparing the recurrence coefficients we see that our polynomials coincide with (rescaled) Meixner-Pollaczeck polynomials with the parameters $\lambda = 1/2$, $\phi = t + \pi/2$.

The explicit expression for our polynomials $P_n(x;t)$ is

$$P_n(x;t) = \frac{n! i^n e^{int}}{\cos^n t} {}_2F_1 \left(\begin{matrix} -n, 1/2 + ix/2 \\ 1 \end{matrix} ; 1 + e^{-2it} \right) \tag{7.4}$$

From the standard formulas for the Meixner-Pollaczeck polynomials [13] we find that that our polynomials $P_n(x;t)$ are orthogonal on whole real axis:

$$\int_{x=-\infty}^{\infty} P_n(x;t) P_m(x;t) W(x;t) dx = h_n(t)\, \delta_{nm}, \tag{7.5}$$

where the weight function $W(x,t)$ is

$$W(x,t) = \frac{\pi\, e^{tx}}{2\cosh(\pi x/2)} \tag{7.6}$$

The weight is well defined provided that t belongs to the admissible interval $-\pi/2 < t < \pi/2$. When t is inside this interval all moments $c_n(t)$ exist and we have the normalization condition

$$\int_{-\infty}^{\infty} W(x;t) dx = \int_{-\infty}^{\infty} \frac{\pi\, e^{tx}}{2\cosh(\pi x/2)} dx = c_0(t) = \frac{1}{\cos t}$$

It is instructive to see how the continuous orthogonality relation (7.5) arises in the limiting case of the discrete-type orthogonality relation (4.5). Indeed, in the trigonometric limit $k \to 0$ we have $K(k) \to \pi/2$ and $K'(k) \to \infty$. So (recall that we assume $e_1 = 2/3, e_2 = e_3 = -1/3$) from (4.3) we see that the grid step $\Delta x(s) = x_{s+1} - x_s$ becomes infinitely small and the sum in rhs of (4.5) becomes an integral (7.5) after appropriate definition of the continuous variable x.

Note also that the special case considered in the previous section (recurrence coefficients given by (6.12), (6.13)) in the limit $k = 0$ corresponds to the formulas (7.1) for $t = \pi/4$:

$$u_n = 2n^2, \quad b_n = 2n + 1.$$

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Consider now more general class of degenerated solutions corresponding to the case when $\beta = 0, \mu_1 = wq/3, \mu_0 = 0$ and q is an arbitrary real parameter. We then have in the hyperbolic limit $(k = 1)$

$$c_0(t) = \frac{\sinh(wt + q)}{\sinh(q)\sinh(wt)} = \frac{1}{2(e^{2q} - 1)} + \frac{2}{1 - e^{-2wt}} \tag{7.7}$$

Introduce the function

$$c_0^{(0)}(t) = \frac{2}{1 - e^{-2wt}} = \frac{e^{wt}}{\sinh(wt)}. \tag{7.8}$$

The function (7.7) differs from $c_0^{(0)}(t)$ only by adding of a term $e^{-q}/\sinh(q)$ not depending on t. We have

$$c_n(t) = c_n^{(0)}(t) = \frac{d^n c_0^{(0)}(t)}{dt^n}, \quad n = 1, 2, \ldots$$

Thus all moments corresponding to functions $c_0^{(0)}(t)$ and $c_0(t)$ coincide apart from zero moments.

Introduce linear functional $\sigma(t)$ and $\sigma^{(0)}(t)$ by their moments

$$\langle \sigma(t), x^n \rangle = c_n(t), \quad \langle \sigma^{(0)}(t), x^n \rangle = c_n^{(0)}(t), \quad n = 0, 1, 2, \ldots$$

We see that the functionals $\sigma(t)$ and $\sigma^{(0)}(t)$ are related as

$$\sigma^{(0)}(t) = \sigma(t) + e^{-q}/\sinh(q)\delta_0, \tag{7.9}$$

where δ_0 is the Dirac delta-functional, corresponding to inserting a unit concentrated mass to the point $x = 0$:

$$\langle \delta_0, x^n \rangle = \delta_{n0}$$

Assume that $\rho^{(0)}(x)$ is the orthogonality measure for the polynomials $P_n^{(0)}(z; t)$ corresponding to the functional $\sigma^{(0)}(t)$:

$$\int_{-\infty}^{\infty} P_n^{(0)}(x; t) P_m^{(0)}(x; t) d\rho^{(0)}(x; t) = h_n^{(0)} \, \delta_{nm}$$

Then the orthogonality measure corresponding to he polynomials $P_n(x; t)$ is

$$\rho(x; t) = \rho^{(0)}(x; t) + e^{-q}/\sinh(q) \, \delta(x)$$

Thus indeed the weight of orthogonality for the polynomials $P_n(x; t)$ is obtained from the corresponding orthogonality weight for the polynomials $P_n^{(0)}(x; t)$ by adding of a concentrated mass $\coth(q)$ at the point $x = 0$.

Now we show that the function $c_0^{(0)}(t)$ given by (7.8) generates a special class of the Meixner polynomials. Indeed, it is elementary verified that two sequences

$$b_n(t) = -\frac{2w(n + 1 + ne^{2wt})}{e^{2wt} - 1}, \quad u_n = \frac{4w^2 n^2 e^{2wt}}{(e^{2wt} - 1)^2}, \quad n = 0, 1, 2, \ldots \tag{7.10}$$

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satisfy the restricted Toda chain equations (1.1) together with the condition $b_0(t) = \dot{c}_0^{(0)}(t)/c_0^{(0)}(t)$. Thus the recurrence coefficients (7.10) correspond to orthogonal polynomials $P_n^{(0)}(z;t)$ having the moments $c_n^{(0)}(t) = d^n c_0^{(0)}(t)/dt^n$. On the other hand, we can easily identify the recurrence coefficients (7.10) with the those for the special class of the Meixner polynomials.

Indeed, the Meixner polynomials $P_n(x;\beta;c)$ have two real parameters β, c and have the recurrence coefficients [13]

$$b_n = \frac{n + (n + \beta)c}{1 - c}, \quad u_n = \frac{c}{(1-c)^2} n(n + \beta - 1) \tag{7.11}$$

Explicitly the Meixner polynomials are expressed in terms of the Gauss hypergeometric function [13]

$$P_n(x;\beta;c) = \kappa_n \, {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix}; 1 - 1/c\right) \tag{7.12}$$

where κ_n is an appropriate normalization factor to provide monicity of the polynomials $P_n(x;\beta;c)$. The Meixner polynomials are orthogonal on the uniform semi-infinite grid [13]:

$$\sum_{s=0}^{\infty} \frac{c^s (\beta)_k}{k!} P_n(k;\beta;c) P_m(k;\beta;c) = h_n \, \delta_{nm} \tag{7.13}$$

Obviously we should have $0 < c < 1$ to provide positivity property of the measure.

Comparing the recurrence coefficients (7.10) with (7.11) we see that $\beta = 1$, $c = e^{-2wt}$ and polynomials $P_n^{(0)}(x;t)$ coincide with the corresponding rescaled Meixner polynomials:

$$P_n^{(0)}(x;t) = (-2w)^n \, P_n(-x/(2w); 1; e^{-2wt}) \tag{7.14}$$

Orthogonality relation for the polynomials $P_n^{(0)}(x;t)$ looks as

$$\sum_{s=0}^{\infty} e^{-2swt} P_n^{(0)}(-2ws;t) P_m^{(0)}(-2ws;t) = h_n^{(0)}(t) \, \delta_{nm} \tag{7.15}$$

To provide positivity of the measure for $t > 0$ we should have $w > 0$. Thus polynomials $P_n^{(0)}(x;t)$ are orthogonal on the uniform grid of the negative real axis.

Return to the polynomials $P_n(z;t)$ corresponding to the function (7.8). The recurrence coefficients for the polynomials $P_n(z;t)$ are obtained from the recurrence coefficients (3.2), (3.1) by the limiting procedure $e_2 \to e_1$:

$$b_n(t) = w(n + 1)\coth(w(n+1)t + q) - wn\coth(wnt + q) - w(2n + 1)\coth(wt) \tag{7.16}$$

and

$$u_n(t) = w^2 n^2 \frac{\sinh(w(n+1)t + q)\sinh(w(n-1)t + q)}{\sinh^2(wnt + q)\sinh^2(wt)} \tag{7.17}$$

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As we already showed, the orthogonality relation for the polynomials $P_n(x;t)$ corresponding to the function (7.8) is obtained from (7.15) by adding of a concentrated mass at the point $x = 0$. Explicitly we have

$$\sum_{s=0}^{\infty} 2e^{-2wst} P_n(-2ws,t) P_m(-2ws,t) + M P_n(0;t) P_m(0;t) = h_n \, \delta_{nm} \tag{7.18}$$

where the value of the mass inserted at $x = 0$ is

$$M = \frac{e^{-q}}{\sinh(q)}.$$

It is assumed that $w, t > 0$ in order to provide convergence of series in lhs of (7.18).

The normalization constants $h_n(t)$ are expressed through the recurrence coefficients (7.17) as

$$h_n(t) = c_0(t) u_1(t) u_2(t) \ldots u_n(t) \tag{7.19}$$

Note that for $n = m = 0$ formula (7.18) gives an obvious identity

$$\sum_{s=0}^{\infty} e^{-2wts} + \frac{e^{-q}}{\sinh(q)} = \frac{\sinh(wt+q)}{\sinh(q)\sinh(wt)} = c_0(t)$$

When $q \to \infty$ we see that $M \to 0$ and $c_0(t) \to \frac{e^{wt}}{\sinh(wt)} = c_0^{(0)}(t)$ hence in this limit the polynomials $P_n(z;t)$ become the ordinary Meixner polynomials (7.14).

The polynomials obtained by an adding of a concentrated mass at the point $x = 0$ of the orthogonality measure for the Meixner polynomials are called the modified Meixner polynomials and were proposed by R.Askey as an interesting object for further investigations [5]. Properties of these polynomials were intensively studied in [3] and [6]. In particular it was shown that these polynomials satisfy difference equations of finite and infinite order.

8. Trigonometric limit

Consider the trigonometric limit when $e_1 = 2/3, e_2 = e_3 = -1/3$. Put $\beta = 0, \mu_0 = 0, \mu_1 = -wq/3$. The recurrence coefficients take the form

$$b_n(t) = w(n+1)\cot(w(n+1)t+q) - wn\cot(wnt+q) - (2n+1)w\cot(wt) \tag{8.1}$$

and

$$u_n(t) = w^2 n^2 \frac{\sin((n+1)wt+q)\sin((n-1)wt+q)}{\sin^2(nwt+q)\sin^2(wt)} \tag{8.2}$$

and the function $c_0(t)$ is

$$c_0(t) = \frac{\sin(wt+q)}{\sin(q)\sin(wt)} \tag{8.3}$$

From the expression (8.2) we see that for real value t, w the coefficient $u_n(t)$ cannot have the same sign for all n. Hence, corresponding measure for orthogonal polynomials is not positive definite.

Nevertheless, there is one interesting special case leading to a positive definite measure on a *finite set* of points on the real axis. Indeed, put $w = 1, q = \pi/2$, then

$$c_0(t) = \cot(t) \tag{8.4}$$

and

$$b_n(t) = -(n+1)\tan((n+1)t) + n\tan(nt) - (2n+1)\cot(t),$$

$$u_n(t) = n^2 \frac{\cos((n+1)t)\cos((n-1)t)}{\cos^2(nt)\sin^2(t)}$$

Assume that

$$t = \tau = \frac{\pi}{2(N+2)}$$

for some positive integer $N = 2, 3, \ldots$. Then it is seen that $u_k > 0$ for $1 \le k \le N$ and $u_{N+1} = 0$. This condition guarantees that the finite set of polynomials $P_0(x; \tau), P_1(x; \tau), \ldots, P_N(x; \tau)$ will be orthogonal on the set of (real) zeros x_s of the polynomial $P_{N+1}(x; t)$:

$$\sum_{s=0}^{N} \rho_s P_n(x_s; \tau) P_m(x_s; \tau) = h_n \, \delta_{nm} \tag{8.5}$$

with postive weights ρ_s.

However explicit expressions for zeros x_s and the weights ρ_s in this case are still unknown.

9. Completely degenerated case. The Krall-Laguerre polynomials

Finally, consider the case when all roots coincide $e_1 = e_2 = e_3 = 0$. Then the elliptic functions are degenerated to simple rational ones: $\wp(z) = 1/z^2$, $\zeta(z) = 1/z$, $\sigma(z) = z$.

Assuming again $w = 1, \beta = 0$ we have

$$c_0(t) = \frac{t + q}{qt} = 1/t + 1/q \tag{9.1}$$

The first term $1/t$ in rhs of (9.1) generates the Laguerre polynomials. Indeed, it is elementary verified that

$$b_n(t) = -\frac{2n+1}{t}, \quad u_n(t) = \frac{n^2}{t^2}$$

is a solution of the restricted Toda chain corresponding to the initial condition $c_0^{(0)}(t) = 1/t$. These recurrence coefficients correspond to the Laguerre polynomials $L_n^{(0)}(-xt)$ [13].

The second constant term in (9.1) describes adding of a concentrated mass to the measure at the endpoint $x = 0$ of the orthogonality interval. We thus obtain that the polynomials $P_n(z;t)$ corresponding to (9.1) coincide with the so-called Krall-Laguerre polynomials (see, e.g. [14] for details). The measure for the latter is obtained by the adding of (an arbitrary) concentrated mass to the point $x = 0$ of the orthogonality interval for the Laguerre polynomials $L_n^{(0)}(x)$. In our case the orthogonality relation looks as

$$\int_{-\infty}^{0} P_n(x;t)P_m(x;t)e^{xt}dx + \frac{P_n(0;t)P_m(0;t)}{q} = h_n(t)\,\delta_{nm} \tag{9.2}$$

The recurrence coefficients are

$$b_n(t) = -\frac{n}{nt+q} + \frac{n+1}{(n+1)t+q} - \frac{2n+1}{t}$$
$$u_n(t) = \frac{n^2}{t^2}\frac{((n-1)t+q)((n+1)t+q)}{(nt+q)^2} \tag{9.3}$$

Remarkably enough that the Krall-Laguerre polynomials belong to a class of 3 families of "non-classical" orthogonal polynomials satisfying the ordinary eigenvalue problem for the linear differential operator of the 4-th order [14].

10. Continued fractions and the Hankel determinants connected with the Jacobi elliptic functions

As we saw, any solution $u_n(t), b_n(t)$ of the restricted Toda chain is generated by the only function $c_0(t)$.

The Stieltjes function $F(z;t)$ for the corresponding orthogonal polynomials $P_n(z;t)$ is given by the Laplace transform

$$F(z;t) = \frac{1}{c_0(t)} \int_0^{\infty} c_0(y+t)\, e^{-yz} dy \tag{10.1}$$

(The factor $1/c_0(t)$ in front of the integral in (10.1) is needed to provide "conventional" asymptotic behavior $F(z;t) = z^{-1} + O(z^{-2})$). On the other hand, it is well known [8] that any Stieltjes function with such asymptotic behavior generates a continued fraction of the Jacobi type (so-called J-type continued fraction):

$$F(z) = \cfrac{1}{z - b_0 - \cfrac{u_1}{z - b_1 - \cfrac{u_2}{z - b_2 - \ldots}}}, \tag{10.2}$$

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where b_n, u_n are corresponding recurrence coefficients for the polynomials $P_n(z; t)$. Thus we can construct explicitly families of continued fractions (10.2) starting from known solution $u_n(t), b_n(t)$ of the Toda chain corresponding to the function $c_0(t)$ given by (3.4).

For a special choice of the parameters β, q, μ_1 we can obtain families of the Jacobi elliptic functions $\mathrm{sn}(t), \mathrm{cn}(t), \mathrm{dn}(t)$ as well as related functions obtained by simplest modular transforms. Thus we can generalize the Stieltjes results (extended and generalized by Milne [16]) who obtained continued fractions corresponding to the Laplace transform of the Jacobi elliptic functions in a special case $t = 0$ of the formula (10.1).

Moreover, from our results it follows a simple formula for the corresponding Hankel determinants $D_n(t)$ defined by (1.15).

Indeed, by (1.10) we have

$$D_n(t) = h_0(t)h_1(t)\dots h_{n-1}, \quad n = 1, 2, 3, \dots \tag{10.3}$$

or, equivalently,

$$D_n(t) = c_0(t)u_1^{n-1}(t)u_2^{n-2}(t)\dots u_{n-2}^2(t)u_{n-1}(t) \tag{10.4}$$

Using explicit formulas (3.2) and (3.1) we find

$$h_n(t) = e^{\mu_1(t+\beta)+\mu_0} n!^2 w^{2n} \frac{\sigma((n+1)w(t+\beta)+q)}{\sigma(nw(t+\beta)+q)\sigma^{2n+1}(wt+\beta)}. \tag{10.5}$$

Note that the parameter μ_0 is inessential, because it doesn't contribute to the recurrence coefficients $b_n(t), u_n(t)$, nevertheless such parameter is convenient when we would like to take $c_0(t)$ coinciding with prescribed Jacobi functions. For the Hankel determinants we have the expression

$$D_n(t) = \kappa_n \frac{\sigma(wn(t+\beta)+q)}{\sigma(q)\sigma^{n^2}(w(t+\beta))} \exp((\mu_1(t+\beta)+\mu_0)n) \tag{10.6}$$

where

$$\kappa_n = 1!^2 2!^2 \dots (n-1)!^2 w^{n(n-1)} \tag{10.7}$$

Consider now 3 special cases corresponding to the basic Jacobi elliptic functions $\mathrm{sn}(t), \mathrm{cn}(t), \mathrm{dn}(t)$. In all these case we can assume that $e_1 - e_3 = 1$. Indeed, the Jacobi elliptic functions $\mathrm{sn}(t, k), \mathrm{cn}(t, k), \mathrm{dn}(t, k)$ depend on the modulus

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}.$$

Hence we can pass from given parameters e_i to the scaling parameters γe_i, $i = 1, 2, 3$ with some nonzero constnat γ. Such transformation doesn't change the Jacobi elliptic functions. Hence we can always can assume that $e_1 - e_3 = 1$.

For the function $\mathrm{sn}(t)$ we put $w = 1, \beta = q = \omega_3$. We then have the recurrence coefficients

$$u_{2n}(t) = 4n^2 k^2 \left(\mathrm{sn}^2(t) - \mathrm{sn}^2(2nt) \right)$$

$$u_{2n+1}(t) = (2n+1)^2 \left(k^2 \mathrm{sn}^2(t) - \frac{1}{\mathrm{sn}^2((2n+1)t)} \right)$$

and

$$b_{2n}(t) = (2n+1)\left\{ Z((2n+1)t + iK') + i\pi/(2K) \right\} - 2nZ(2nt) - (4n+1)Z(t)$$

$$b_{2n+1}(t) = (2n+2)Z((2n+2)t, k) - (2n+1)(Z((2n+1)t + iK', k) + i\pi/(2K)) - (4n+3)Z(t, k)$$

For the function $\mathrm{cn}(t)$ we put $w = 1$, $\beta = \omega_3$, $q = \omega_2$. We then have the recurrence coefficients

$$u_{2n}(t) = 4n^2 k^2 \left(-\mathrm{cn}^2(t) + k'^2 \frac{\mathrm{sn}^2(2nt)}{\mathrm{dn}^2(2nt)} \right)$$

$$u_{2n+1}(t) = (2n+1)^2 \left(-\mathrm{dn}^2(t) - k'^2 \frac{\mathrm{sn}^2((2n+1)t)}{\mathrm{cn}^2((2n+1)t)} \right)$$

and

$$b_{2n}(t) = (2n+1)\left\{ Z((2n+1)t, k) - \frac{\mathrm{sn}((2n+1)t, k)\mathrm{dn}((2n+1)t, k)}{\mathrm{cn}((2n+1)t, k)} \right\} - 2nZ(2nt + K, k) - (4n+1)Z(t, k)$$

$$b_{2n+1}(t) = (2n+2)Z((2n+2)t, k) + K) +$$
$$(2n+1)\left\{ -Z(2n+1)t, k) + \frac{\mathrm{sn}((2n+1)t, k)\mathrm{dn}((2n+1)t, k)}{\mathrm{cn}((2n+1)t, k)} \right\} -$$
$$-(4n+3)Z(t, k)$$

For the function $\mathrm{dn}(t)$ we put $w = 1$, $\beta = \omega_3$, $q = \omega_1$. We then have the recurrence coefficients

$$u_{2n}(t) = 4n^2 \left(-\mathrm{dn}^2(t) - k'^2 \frac{\mathrm{sn}^2(2nt)}{\mathrm{cn}^2(2nt)} \right)$$

$$u_{2n+1}(t) = (2n+1)^2 k^2 \left(-\mathrm{cn}^2(t) + k'^2 \frac{\mathrm{sn}^2((2n+1)t)}{\mathrm{dn}^2((2n+1)t)} \right)$$

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and

$$b_{2n}(t) = (2n + 1)Z((2n + 1)t + K, k) - 2n \left\{ Z(2nt, k) - \frac{\text{sn}(2nt)\text{dn}(2nt)}{\text{cn}(2nt)} \right\} - (4n + 1)Z(t, k)$$

$$b_{2n+1}(t) = (2n + 2) \left\{ Z((2n + 2)t, k) - \frac{\text{sn}((2n + 2)t)\text{dn}((2n + 2)t)}{\text{cn}((2n + 2)t)} \right\} -$$
$$(2n + 1)Z(2nt + K, k) - (4n + 3)Z(t, k)$$

11. Concluding remarks

The function $c_0(t)$ given by (3.4) is closely related with simplest solutions of the Lamé equation.

Indeed, consider the Lamé equation in the form [2]

$$\frac{d^2 y}{du^2} = \{n(n + 1) \, \wp(u) + l\} \, y \tag{11.1}$$

In case if n is a positive integer one can construct explicit solutions of the Lame equation in the form [2]

$$\phi(u) = e^{\lambda u} \frac{\sigma(u - a_1) \dots \sigma(u - a_n)}{\sigma^n(u)} \tag{11.2}$$

where the constants $\lambda, a_1, \dots, a_n$ can be determined from the Lamé equation. In the simplest nontrivial case $n = 1$ we have

$$\phi(u) = \frac{\sigma(u + a)}{\sigma(u)} \, \exp(-\zeta(a)u) \tag{11.3}$$

where the parameter a is related with the spectral parameter l of the Lamé equation by the transcendental equation $\wp(a) = l$. In case if $a \neq \omega_k$, $k = 1, 2, 3$ we have the second linearly independent solution of the Lamé equation in the form

$$\phi(u) = \frac{\sigma(u - a)}{\sigma(u)} \, \exp(\zeta(a)u) \tag{11.4}$$

We see that our function $c_0(t)$ (3.4) coincides with the Lamé solution (11.3) when $\mu_1 = -\zeta(a)$.

This means that if the Stieltjes function $F(z; t)$ is the Laplace transform of the solution of the Lamé equation

$$F(z; t) = \int_0^\infty \left\{ \frac{\sigma(u + a + t)}{\sigma(u + t)} \, e^{-(\zeta(a)+z)u} \right\} du \tag{11.5}$$

then corresponding orthogonal polynomials $P_n(z; t)$ will have the recurrence coefficients given by (3.2) and (3.1) where $w = 1$, $\beta = 0, q = a, \mu_1 = -\zeta(a)$.

In [10], [11] some explicit continued fractions connected with the Lamé solutions (11.3) and (11.2) were announced without any proof or even idea of proof. The authors of [10], [11] considered Stieltjes functions of kind of (11.5) but the argument of these function was $w = \wp(a)$ instead of z. This leads to explicit continued fractions which do not resemble presented in the present paper. It would be desirable to connect results in [10] and [11] with our ones.

Another possible generalization consists in passing to the associated polynomials. Indeed, we considered here only solutions for the restricted Toda chain, i.e. under the condition $u_0 = 0$. Nevertheless, solutions (3.2), (3.1) can be easily extended to the non-restricted case if one replace n with $n + c$ in corresponding formulas, where c is an arbitrary constant not depending on t. Then we obtain solution of the nonrestricted Toda chain if $c \neq 0, \pm 1, \pm 2, \ldots$. Such replacement $b_n \to b_{n+c}$, $u_n \to u_{n+c}$ is well known and leads to replacing of the orthogonal polynomials $P_n(z;t)$ with their *c-associated* polynomials. Valent already considered [23], [24] the c-associated polynomials corresponding to the Stieltjes-Carlitz polynomials. He was able to find an explicit orthogonality measure in some special cases. In our case (i.e. for $t \neq 0$) the corresponding analysis seems to be much more complicated.

Nevertheless, there is a simple special case of the associated polynomials for $t \neq 0$ which leads again to solutions of the restricted Toda chain.

Indeed, take $c = 1$ in formulas (3.1) and (3.2) and then put $q = 0$, $w = 1, \beta = 0, \mu_1 = 0$. We obtain the recurrence coefficients

$$b_n(t) = (n + 2)\zeta((n + 2)t) - (n + 1)\zeta((n + 1)t) - (2n + 3)\zeta(t) \tag{11.6}$$

and

$$u_n(t) = (n + 1)^2 \left(\wp(t) - \wp((n + 1)t)\right) = \frac{\sigma(nt)\sigma((n + 2)t)}{\sigma^2(t)\sigma^2((n + 1)t)} \tag{11.7}$$

It is seen from (11.7) that $u_0(t) = 0$, hence we deal again with a solution of the restricted Toda chain. From (11.6) we have

$$b_0(t) = 2\zeta(2t) - 4\zeta(t) = \dot{c}_0(t)/c_0(t)$$

whence

$$c_0(t) = \dot{\wp}(t), \tag{11.8}$$

where we used the identity [26]

$$\zeta(2z) = \zeta(z) + \frac{\wp''(z)}{2\wp'(z)}$$

Thus we obtained that the function $c_0(t) = \dot{\wp}(t)$ generates another solution of the restricted Toda chain described by formulas (11.6), (11.7). This solution was already presented by Chudnovsky brothers in [12] The corresponding orthogonal polynomials $P_n(z;t)$ seems not to possess

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positivity property for the Hankel determinants $D_n(t)$, hence their measure will not be positive on the real axis.

Nevertheless, if $t = 2\omega_1/(N + 2)$ with some positive integer N we have $u_N(t) = 0$. This means that polynomials $P_n(z; t)$ will be orthogonal on a finite set of points x_s:

$$\sum_{s=0}^{N-1} w_s P_n(x_s; t) P_m(x_s; t) = h_n(t)\, \delta_{nm}, \quad n, m = 0, 1, \dots, N - 1 \tag{11.9}$$

where x_s are roots of the polynomial $P_N(x)$:

$$P_N(x_s) = 0, \quad s = 0, 1, \dots, N - 1$$

Assume that all the roots x_s are simple. Then the discrete weight function w_s can be presented in the form [8]

$$w_s = \frac{h_{N-1}}{P_{N-1}(x_s) P_N'(x_s)} \tag{11.10}$$

Moreover, for the canonical choice of the parameters e_i, i.e. $e_3 < e_2 < e_1$ we have $u_n > 0$, $n = 1, 2, \dots, N - 1$ and hence all the weights will be positive $w_s > 0$, $s = 0, 1, \dots, N - 1$. Finding explicit expression for x_s and w_s is an interesting open problem.

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